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Homeomorphims Between Cantor Sets

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1 Preliminaries

Definition 1. *Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ be intervals that are either both closed or both open in $\mathbb{R}$, with $a_1 \neq b_1$. Then we define*

$$\Lambda(I_1, I_2) : I_1 \to I_2$$
$$\Lambda(I_1, I_2) : x \mapsto (x - a_1)\frac{b_2 - a_2}{b_1 - a_1} + a_2$$

Lemma 1. *Let I_1, I_2 be intervals as above. Then $\Lambda(I_1, I_2)$ is increasing. If $b_2 \neq a_2$, it is strictly increasing.*

Proof. Let $x_1, x_2 \in I_1$ such that $x_2 > x_1$. Then

$$\begin{aligned}
\Lambda(I_1, I_2)(x_2) - \Lambda(I_1, I_2)(x_1) &= (x_2 - a_1)\frac{b_2 - a_2}{b_1 - a_1} + a_2 - (x_1 - a_1)\frac{b_2 - a_2}{b_1 - a_1} + a_2 \\
&= (x_2 - a_1)\frac{b_2 - a_2}{b_1 - a_1} - (x_1 - a_1)\frac{b_2 - a_2}{b_1 - a_1} \\
&= (x_2 - x_1)\frac{b_2 - a_2}{b_1 - a_1} \geq 0
\end{aligned}$$

If $b_2 > a_2$, this last inequality is strict. □

Lemma 2. *Let I_1, I_2 be intervals as above. Then if $b_2 \neq a_2$, $\Lambda(I_1, I_2)$ is invertible (in fact, a homeomorphism). Furthermore, $\Lambda^{-1}(I_1, I_2) = \Lambda(I_2, I_1)$.*

Proof. We prove this directly: let $x \in I_1$, $y = \Lambda(I_1, I_2)(x)$. Then

$$y = (x - a_1)\frac{b_2 - a_2}{b_1 - a_1} + a_2 \implies (y - a_2)\frac{b_1 - a_1}{b_2 - a_2} + a_1 = x$$

In other words, $x = \Lambda^{-1}(I_1, I_2)(y) = \Lambda(I_2, I_1)(y)$. □

Definition 2. *Let D be a closed interval in $\mathbb{R}$. Then $\{D_i\}_1^N$ is a* ordered partition *of D if:*

1. *Each D_i is either open or closed.*
2. *$\bigcup_{i=1}^N D_i = D$ and for $i \neq j$, $D_i \cap D_j$ has at most one point.*
3. *For $i = 1, \ldots, N-1$, the right endpoint of D_i is the left endpoint of D_{i+1}.*

Lemma 3. *Let D, R be closed intervals in $\mathbb{R}$. Let $\{D_i\}_1^N$, $\{R_i\}_1^N$ be ordered partitions of D and R, with each $|D_i| > 0$. Let $\Lambda : D \to R$ be defined as*

$$\Lambda(x) = \Lambda(D_i, R_i)(x) \qquad \textit{for } x \in D_i$$

be a piecewise linear function. Then Λ is well-defined and increasing. If all $|R_i| > 0$, then Λ is strictly increasing.

Proof. Suppose $x \in D_i \cap D_j$ for $i > j$. Then x must be the right endpoint of D_i and the left endpoint of D_j, so by Definition 1, $\Lambda(D_i, R_i) = \Lambda(D_j, R_j)$. So Λ is well-defined.

Then, by Lemma 1, Λ is increasing (or strictly increasing) on each interval. $\square$

Lemma 4. *Let D, R be closed intervals in $\mathbb{R}$. Let $\{D_i\}_1^N$, $\{D_i'\}_1^M$ be ordered partitions of D such that $|D_i| > 0$ and $|D_i'| > 0$ for all i. Let $\{R_i\}_1^N$, $\{R_i'\}_1^M$, be ordered partitions of R. Let $\Lambda' : D \to R$, $\Lambda : D \to R$ be defined as*

$$\Lambda(x) = \Lambda(D_i, R_i)(x) \qquad \Lambda'(x) = \Lambda(D_i', R_i')(x) \qquad \textit{for } x \in D_i$$

be piecewise linear functions. Then $|\Lambda(x) - \Lambda'(x)| \leq |R|$ for all $x \in D$.

Proof. By Lemma 3 we have that Λ and Λ' are strictly increasing. Let d_1, d_2 be the endpoints of D, r_1, r_2 the endpoints of R. By construction $\Lambda(d_1) = \Lambda'(d_1) = r_1$ and $\Lambda(d_2) = \Lambda'(d_2) = r_2$. Thus, for any $x \in D$,

$$\Lambda(d_1) = r_1 \leq \Lambda(x) \leq r_2 = \Lambda(d_2) \text{ and } \Lambda'(d_1) = r_1 \leq \Lambda'(x) \leq r_2 = \Lambda'(d_2)$$

Assume WLOG that $\Lambda'(x) \leq \Lambda(x)$. Then

$$\Lambda(x) - \Lambda'(x) \leq r_2 - \Lambda'(x) \leq r_2 - r_1 = |R| \qquad \square$$

Definition 3. *We define a* Cantor-like sequence *to be a sequence $\{P^i\}_0^\infty$ of ordered partitions of $[0,1]$ such that P^0 contains the single interval $[0,1]$, and for all $i \geq 1$, P^i is defined from P^{i-1} by replacing each closed interval $[a,b]$ by the three intervals*

$$\left[a, \frac{b+a-\ell_i}{2}\right] \qquad \left(\frac{b+a-\ell_i}{2}, \frac{b+a+\ell_i}{2}\right) \qquad \left[\frac{b+a+\ell_i}{2}, b\right]$$

with $\ell_i > 0$ chosen such that

$$\ell_1 + 2\ell_2 + 4\ell_3 + \cdots + 2^{i-1}\ell_i + \cdots \leq 1$$

Definition 4. *Let $\{P^i\}_0^\infty$ be a Cantor-like sequence. Define a new sequence $\{C^i\}_1^\infty$ by*

$$C^i = \bigcup_{j=0}^{\infty} P^i_{2j} \qquad i = 0, 1, \ldots$$

so that each C^i is the union of the closed sets in P^i. Then

$$\mathcal{C} = \bigcap_{i=1}^{\infty} C_i$$

is a Cantor-like set.

Remark. Choosing $\ell_i = 1/3^i$ gives us the traditional "remove one-third" Cantor set. Choosing $\ell_i = 1/2^{2i}$ gives us the Smith–Volterra–Cantor "fat Cantor" set.

2 Facts About Cantor-Like Sets

Let $\mathcal{C}$ be an arbitrary Cantor-like set, $\{C^i\}$ and $\{P^i\}$ as in Definition 4. We notice:

- The size of the intervals in C^i tend to 0 as $i \to \infty$. This is because there are 2^i such intervals, all of the same length, whose total length can never exceed 1.
- The endpoints of every interval in every C^i are in $\mathcal{C}$. This is by construction.

Further, $\mathcal{C}$ has the following properties:

- *Closed.* It is defined as the countable intersection of closed sets.
- *Perfect.* Let $x \in \mathcal{C}$ and let $\epsilon > 0$. Then for every C^i, $x \in C^i_j$ for some j. As $i \to \infty$, the size of these C^i_j tends to 0, so by choosing the right endpoint x_i of C^i_j, we have a sequence $\{x_i\}$ such that $x_i \to x$. Thus x is a limit point of $\mathcal{C}$.
- *No interior points.* Choose the sequence $\{x_i\}$ from the previous claim, and define $\{y_i\}$ by $y_i = x_i + \ell_i/2$. Then $y_i \notin \mathcal{C}$ (it is removed in the ith step) but since $\ell_i \to 0$, we still have $y_i \to x$. Thus x is not an interior point.
- *Uncountability.* We will derive a surjection from $\mathcal{C}$ to $[0, 1]$.

Theorem 1. *Let $\{\ell_i\}$ be as in Definition 4. Then $m(\mathcal{C}) = 1 - \sum_1^\infty 2^{i-1}\ell_i$.*

Proof. As the intervals in each C^i are disjoint, we have that

$$1 = m([0,1]) = \sum_{j=0}^{2^i} m(P^i_j) = \sum_{j=0}^{2^{i-1}} m(P^i_{2j}) + \sum_{j=0}^{2^{i-1}} m(P^i_{2j+1}) = m(C^i) + \sum_{j=1}^{i} 2^{j-1}\ell_j$$

using finite additivity and the definition of C^i and ℓ_i. Therefore, for each $i \in \mathbb{N}$,

$$m(C^i) = 1 - \sum_{j=1}^{i} 2^{j-1}\ell_j$$

Finally, because $C^i \supset C^{i+1}$ for each i, by Corollary 3.3 in chapter 1 of Stein/Shakarchi,

$$m(\mathcal{C}) = m\left(\bigcap_{i=1}^{\infty} C^i\right) = \lim_{i\to\infty} m(C^i) = 1 - \sum_{j=1}^{\infty} 2^{j-1}\ell_j$$

$\square$

Remark. For the traditional Cantor set, $\ell_i = (1/3)^i$ and this limit evaluates to 0, as expected. But for the fat Cantor set, we get 1/2, though all the usual properties of the set hold!

3 The Cantor-Lebesgue Function

Define the following sequence $\{Q^i\}_0^\infty$ of ordered partitions of $[0,1]$: let Q^0 contain only $[0,1]$. Then for each $i \geq 1$, construct Q^i from Q^{i-1} by replacing each nontrivial interval $[a,b]$ by the intervals

$$\left[a, \frac{a+b}{2}\right] \qquad \left[\frac{a+b}{2}, \frac{a+b}{2}\right] \qquad \left[\frac{a+b}{2}, b\right]$$

Notice that each Q^i covers $[0,1]$ and contains exactly as many intervals as the corresponding P^i from Definition 4.

Now, let $\mathcal{C}$, $\{P^i\}_0^\infty$ be as in Definition 4. Define a sequence of functions $\{\phi_i\}_0^\infty$ by

$$\begin{aligned} &\phi_i : [0,1] \to [0,1] \\ &\phi_i : x \mapsto \Lambda\left(\overline{P_j^i}, Q_i^i\right)(x) \qquad x \in P_j^i \end{aligned}$$

We define the *Cantor-Lebesgue function*:

$$\phi : [0,1] \to [0,1] \qquad \phi(x) = \lim_{n\to\infty} \phi_n(x)$$

Theorem 2. *The Cantor-Lebesgue function is continuous.*

Proof. Let ϕ, $\mathcal{C}$, $\{C^i\}_0^\infty$, $\{P^i\}_0^\infty$, $\{Q^i\}_0^\infty$ and $\{\phi^i\}_0^\infty$ be as above.

By construction, ϕ_i is continuous for all i. We need to show that the limit is continuous. Let $n, m \in \mathbb{N}$, $m > n$. Let $x \in [0,1]$, so that $x \in P_i^n$ for some i. We have two cases:

- $P_i^n = (a,b)$. As this is an open interval, it exists unchanged in P^m, as does its image in Q^m. It follows that $|\phi_n(x) - \phi_m(x)| = 0$.
- $P_i^n = [a,b]$. Then after $m-n$ steps in the Cantor-like set construction, $P_i^n = \bigcup_{j=j_1}^{j_2} P_j^m$ for some j_1, j_2, and $\phi_n(P_i^n) = \bigcup_{j=j_1}^{j_2} \phi_m(P_j^m)$. By Lemma 1, $|\phi_n(x) - \phi_m(x)| \leq |P_i^n|$.

The size of closed $|P_i^n|$ decreases to 0 (as there are 2^n of them, each of which is the same size, and the total length is ≤ 1) as n tends to infinity. Therefore, for any $\epsilon > 0$, we can choose some n sufficiently large so that for all $m > n$,

$$|\phi_n(x) - \phi_m(x)| \leq |P_i^n|$$

This is the Cauchy criterion, and we know from Math 320 that this gives us uniform convergence to ϕ, and the limit of a uniformly converging sequence of continuous functions is itself continuous. $\square$

Notice that if $x \notin \mathcal{C}$, some neighborhood of x will be mapped to a trivial interval. That is, the "holes" in the Cantor-like set will be mapped to constant functions. As a consequence, if $\mathcal{C}$ is the traditional Cantor set of measure 0, ϕ is a continuous increasing function with derivative 0 almost everywhere!

Remark. As $\phi(0) = 0$, $\phi(1) = 1$, the IVT gives us that ϕ is surjective.

Remark. The restriction of ϕ to $\mathcal{C}$ has $[0, 1]$ as its image, since every number x *not* in $\mathcal{C}$ has $\phi(x)$ copied from its nearest endpoints.

4 A Most Suprising Homeomorphism

Now, let $\mathcal{C}$ and $\mathcal{C}'$ be two Cantor-like sets, and $\{P^i\}_1^\infty$, $\{P'^i\}_1^\infty$ the sequences of ordered partitions used in Definition 4 to build them. Proceeding as before, we write

$$\Phi_i : [0, 1] \to [0, 1]$$
$$\Phi_i : x \mapsto \Lambda(P^i_j, P'^i_j)(x) \qquad x \in P^i_j$$

We define the following function:

$$\Phi : [0, 1] \to [0, 1] \qquad \Phi(x) = \lim_{n\to\infty} \phi_n(x)$$

By exactly the same proof as above, Φ is continuous. Furthermore,

Theorem 3. *Φ is strictly increasing.*

Proof. By Lemma 1, ϕ_i is strictly increasing for all i. As the endpoints of every interval in every C^i are fixed by ϕ_n, $n > i$, it follows that ϕ is strictly increasing on the set of interval endpoints in $\mathcal{C}$.

Let $x, y \in [0, 1]$, $x < y$. Then we have two cases:

1. If both x and y are in $\mathcal{C}$, after some number N of iterations, the intervals in C^i, $i > N$, will have length less than $y - x$. Then y and x will lie in different intervals. Let x_1 be the right endpoint of x's interval, y_1 the left endpoint of y's. We then have that for all $i > N$,
$$\phi_i(x) \leq \phi_i(x_1) = \phi(x_1) < \phi(y_1) = \phi_i(y_1) \leq \phi_i(y)$$
and by taking $i \to \infty$, $\phi(x) < \phi(y)$.

2. Otherwise, at least one of x, y is not in $\mathcal{C}$. Assume WLOG that $x \notin \mathcal{C}$. Then after some number of interations N, $\phi_i(x)$ will be fixed for all $i > N$. If $y \notin \mathcal{C}$, $\phi_i(y)$ will also be fixed, so we can use Lemma 1 to see that $\phi(x) = \phi_i(x) < \phi_i(y) = \phi(y)$.

   Otherwise, $y \in \mathcal{C}$, so choose e to be an endpoint between x and y. Then after some number of iterations, $\phi(e)$ is fixed, so $\phi_i(x) = \phi(x) < \phi(e) = \phi_i(e) \leq \phi_i(y)$. Take $i \to \infty$ to see that $\phi(x) < \phi(y)$.

□

Remark. As a conseqeunce of the above, Φ is bijective.