Proof of Van Der Warden's Theorem

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- Let χ be an r-coloring of the interval $[1, MM']$.
- Define $\overline{\chi} : [1, M'] \to [1, r^M]$ as follows:

$$\overline{\chi}(k_1) = \overline{\chi}(k_2) \Leftrightarrow \chi(k_1 M - i) = \chi(k_2 M - i),$$

for each $i \in [0, M-1]$.

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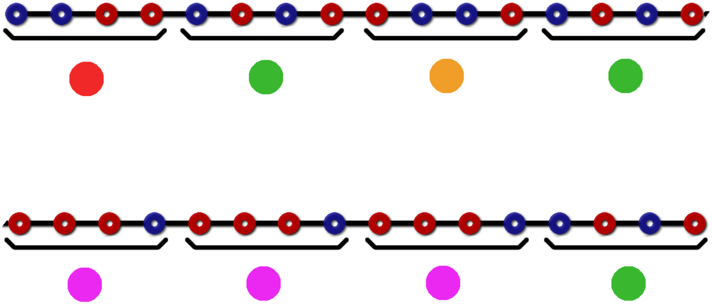
- Then the coloring $\overline{\chi}$ is constructed as:



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- This gives a sequence, $I_0, \ldots, I_{\ell-1}$, of ℓ sub-intervals of length M in $[1, MM']$ each of which is colored the same under χ.

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- Each sub-interval is of the form $I_x := [M(a' + x - 1) + 1, M(a' + x)]$, with $x \in X_0$.

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- This gives a sequence, $I_0, \ldots, I_{\ell-1}$, of ℓ sub-intervals of length M in $[1, MM']$ each of which is colored the same under χ.
- Each sub-interval is of the form $I_x := [M(a' + x - 1) + 1, M(a' + x)]$, with $x \in X_0$.
- Consider I_0. By the induction hypothesis, there exist $a, d_2, \ldots, d_{m+1} \in \mathbb{N}$ such that

$$a + \sum_{i=2}^{m+1} x_i d_i \in I_0, \qquad \chi\left(a + \sum_{i=2}^{m+1} x_i d_i\right) \equiv \text{const}$$

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- We will consider two cases: when $x = \ell$, and when $x < \ell$.

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$$\chi\left(a + \sum_{i=1}^{m+1} x_i d_i\right) = \chi\left(a + \sum_{i=2}^{m+1} x_i d_i\right),$$

and so χ is constant on each $X_j \subset [0, \ell]^{m+1}$, with $j \in [0, m+1]$.

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- Thus the only elements we need to worry about come from $X_{m+1} = \{(\ell, \ldots, \ell)\}$.
- It is clear that χ must take a unique value on X_{m+1}, from which the result follows.

From $S(l, m)$ to $S(l+1, 1)$

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- Now, we show that if statement $S(l, m)$ is true for some l, and *all* values of m, then statement $S(l+1, 1)$ holds.
- In this way, we increase the maximum length of arithmetic progressions that are guaranteed to exist for an r-coloring of the natural numbers.

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- So, choose some r, let $N = N(l, r, r)$, and let χ be an r-coloring of $[1, N]$.
- Then there exist numbers a, $d_1, \ldots, d_r$ such that

$$\chi(a + x_1 d_1 + x_2 d_2 + \cdots + x_r d_r)$$

is constant on each l-equivalence class X_i.

- For each $i = 1, 2, \ldots, r$, define the sum

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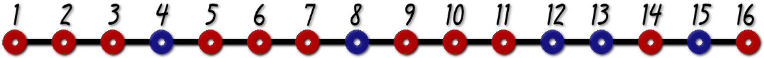
- Choose two specific s_i's, say, s_L and s_H, such that

$$\chi(a + ls_L) = \chi(a + ls_H)$$

Also, suppose $L < H$.

An Example



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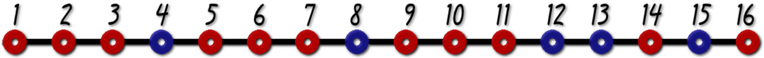

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- For each item (x_1, x_2) in X_0, $\chi(a + d_1 x_1 + d_2 x_2)$ is red. Examples:

$$\begin{aligned} &\text{for } (x_1, x_2) = (1, 2), \quad a + d_1(1) + d_2(2) = 7 \\ &\text{for } (x_1, x_2) = (2, 2), \quad a + d_1(2) + d_2(2) = 11 \end{aligned}$$

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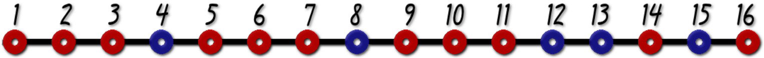

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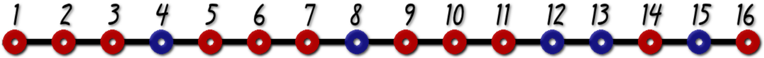

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- Similarly, for each (x_1, x_2) in X_1, $\chi(a + d_1x_1 + d_2x_2)$ is blue.
- Our s_i's are:

$$s_1 = d_1 + d_2 = 5 \qquad s_2 = d_2 = 1 \qquad s_3 = 0$$

The General Claim

- Now, we are ready to show $S(l+1,1)$. This statement is simple, since there is only one nontrivial l-equivalence class:

$$X_0 = \{0, 1, \ldots, l\}$$

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for all $x \in X_0$.

- We claim this works for

$$a' = a + ls_H$$

$$d' = s_L - s_H$$

The Proof

- Suppose that $x < l$. We will show that $\chi(a' + d'x)$ is the same as $\chi(a' + d'l)$. Specifically,

$$\begin{aligned}
\chi(a' + d'x) &= \chi(a + s_H l + (s_L - s_H)x) && (1)\\
&= \chi(a + s_H l + (s_L - s_H)0) && (2)\\
&= \chi(a + s_H l) && (3)\\
&= \chi(a + s_L l) && (4)\\
&= \chi(a + s_H l + (s_L - s_H)l) &&\\
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&= \chi(a + s_H l + (s_L - s_H)l) &&\\
&= \chi(a' + d'l) &&
\end{aligned}$$

- There are two tricks here: getting from (1) to (2), and getting from (3) to (4).

$$\chi(a + s_H l + (s_L - s_H)x) = \chi(a + s_H l + (s_L - s_H)0)$$

$$\chi(a + s_H l + (s_L - s_H)x) = \chi(a + s_H l + (s_L - s_H)0)$$

is really saying that the following are equal:

$$\chi(a + d_L x + \cdots + d_{H-1} x + d_H l + \cdots + d_r l)$$

$$\chi(a + d_L 0 + \cdots + d_{H-1} 0 + d_H l + \cdots + d_r l)$$

$$\chi(a + s_H l + (s_L - s_H)x) = \chi(a + s_H l + (s_L - s_H)0)$$

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This is true because our choice of d_i's; specifically, since the vectors

$$(\underbrace{0, \cdots, 0}_{L-1 \text{ times}}, \underbrace{x, \cdots, x}_{H-L \text{ times}}, l, \cdots l) \text{ and } (\underbrace{0, \cdots, 0}_{L-1 \text{ times}}, \underbrace{0, \cdots, 0}_{H-L \text{ times}}, l, \cdots l)$$

are both in the same l-equivalence class of $[0, l]^r$.

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- We then showed the existence of numbers a', d' such that $\chi(a' + d'x)$ was constant on the set $x \in \{0, 1, \ldots, l\}$.

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- We then showed the existence of numbers a', d' such that $\chi(a' + d'x)$ was constant on the set $x \in \{0, 1, \ldots, l\}$.
- Since this set is the only nontrivial l-equivalence class when considering $S(l + 1, 1)$, the existence of a' and d' gives the result!

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- Since this set is the only nontrivial l-equivalence class when considering $S(l + 1, 1)$, the existence of a' and d' gives the result!
- Therefore, given $S(l, m)$ for all $m \geq 1$, we have $S(l + 1, 1)$.

Putting it all Together

- Angela showed that $S(1,1)$ is true, and Navid showed that if $S(l,1)$ is true, then $S(l,m)$ is true for all $m \geq 1$.

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- Together, these show that $S(l, m)$ is true for all $l \geq 1$, $m \geq 1$.
- Finally, as Angela showed, the specific case $S(l, 1)$ is van der Waerden's theorem!

Thank you for listening. We are:

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Julian Wong
Angela Guo
Navid Alaei
Andrew Poelstra

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This presentation was part of a course at SFU taught by:

Veselin Jungic
Tom Brown
Hayri Ardal

Additional Resources

- B. L. van der Waerden, How the proof of Baudet's conjecture was found, in Studies in Pure Mathematics (Presented to Richard Rado), 251-260, Academic Press, London, 1971
- A.Y. Khinchin, Three Pearls of Number Theory, Garylock Press, Rochester, N. Y., 1952
- Two other classical Ramsey-type theorems: Schur's Theorem and Rado's Single Equation Theorem