Rainbow AP Data

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Definition 1. Fix integers $r \geq 1$, $k \geq 1$. Suppose some number N exists such that every r-coloring of $[1, N]$ contains either a monochromatic k-term arithmetic progression, or a "rainbow AP", i.e., a r-term arithmetic progression with one term of each color.

We denote the least such N by $\rho(k; r)$.

Definition 2. With r and k as before, suppose N exists so that every r-coloring contains a k-term *double* AP, or a rainbow AP. We denote the least such N by $\rho^*(k; r)$.

We notice that any r-coloring of a set is also a $(r+1)$-coloring which never uses the $(r+1)$th color. When considered this way, it is necessarily rainbow-AP-free, since rainbow AP's use every color. We therefore obtain the following relations:

$$w(k; r) \leq \rho(k; r+1)$$

$$w^*(k; r) \leq \rho^*(k; r+1)$$

In particular the existence of ρ^* numbers implies the existence of w^*. The converse is also clear ("every coloring has a double AP" implies "every coloring has a double AP or rainbow AP"), so we get the following theorem:

Theorem 1. $\rho^*(k; r)$ exists for all $k, r \geq 1$ iff $w^*(k; r)$ does.

Proof. By the above reasoning,

$$\rho(k; r) \leq w(k; r) \leq \rho(k; r+1) \leq w(k; r+1)$$

$$\rho^*(k; r) \leq w^*(k; r) \leq \rho^*(k; r+1) \leq w^*(k; r+1)$$

$\square$

We also notice that for all k, $\rho(k, 1) = \rho^*(k, 1) = 1$, and for all r, $\rho(1, r) = \rho^*(1, r) = 1$, so we'll ignore these trivial cases.

Also, when $k = 2$, every r-coloring of $[1, r]$ either contains two elements the same color (and thus a 2-AP and 2-DAP), or all elements different colors (and thus a rainbow AP). We conclude that $\rho^*(2, r) = \rho(2, r) = r$.

Similarly, when $r = 2$, every 2-coloring of $[1, k]$ either contains a k-AP (everything is one color) or a rainbow AP (two things are each a different color). So $\rho^*(k, 2) = \rho(k, 2) = k$.

By brute-force searching, I have obtained the following additional ρ values:

	rho				
k\r	3	4	5	6	7
3	9	57	>84	>152	>164
4	53	>125	>161	>308	
5	>292	>226	>607		
6	>163	>429			
7	>195				

	rho*				
k\r	3	4	5	6	7
3	17	>295	>2036	>4256	>6211
4	>3416	>63500			
5	>77500				
6					
7					