On the Existence of Double 3-Term Arithmetic Progressions

Andrew Poelstra

IRMACS Ramsey Theory Working Group
Department of Mathematics
Simon Fraser University

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- An **arithmetic progression** is a finite subsequence $x_{i_1}, x_{i_2}, \cdots, x_{i_N}$ with a constant difference d between consecutive entries.
- For example, if $2, 4$ and 6 showed up in our sequence, this would be an arithmetic progression. Ditto for 1, 4, 7 and 10.

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1, 7, 15, 19, 25, 26, 32, 39, 44, 47,

52, 59, 66, 73, 75, 79, 87, 89, 94, 99

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- The first few terms are

$$2_4, 1_4, 3_4, 20_4, 22_4, 21_4, 23_4, 100_4, 102_4, 101_4, 103_4, 120_4, \ldots$$

or in base 10,

$$2, 1, 3, 8, 10, 9, 11, 4, 6, 5, 7, 12, \ldots$$

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- This sequence was published by A. F. Sidorenko in 1988.

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- This makes 3-term AP's easier (and faster!) to compute with.

One last definition:

- A **3-coloring** of an interval $[1, N]$ is a partitioning of the interval into 3 disjoint cells, or colors, as

$$\{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_m\} \cup \{z_1, z_2, \ldots, z_\nu\}$$

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  (In general, an r-**coloring** is the same, except with r cells.)

- For example, we might 3-color the interval $[1, 10]$ as

$$\{1, 3, 6, 10\} \cup \{2, 5, 7, 9\} \cup \{4, 8\}$$

  Graphically,

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- In 1927, B. L. van der Waerden (1903-1996) proved this theorem. It had been conjectured (in a less general form) by Baudet (1891-1921) some years earlier.
- It is one the earliest theorems in Ramsey Theory. Together with Schur's Theorem and Ramsey's Theorem, it forms the basis of the field.

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- The six known non-trivial values are:

$$\begin{array}{ll} \mathrm{w}(2,\,3) = 9 & \mathrm{w}(2,\,4) = 35 \\ \mathrm{w}(2,\,5) = 178 & \mathrm{w}(2,\,6) = 1132 \\ \mathrm{w}(3,\,3) = 27 & \mathrm{w}(4,\,3) = 76 \end{array}$$

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- To contrast, the current best upper bound for a given r and k, published by Timothy Gowers in 1998, is

$$w(r,k) \leq 2^{2^{r^{2^{2^{k+9}}}}}$$

Ron Graham has a $1000 prize waiting for anyone who can prove

$$w(2, k) \leq 2^{k^2}$$

Let's modify the statement of van der Waerden's theorem, to ask

- *Given some $r \geq 2$, $k \geq 2$, can we find a number w^* such that every r-coloring of $[1, w^*]$ contains a monochromatic k-term* **double** *AP?*

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- Here are the known non-trivial values of $w^*(r, k)$:

$$w^*(2, 3) = 17$$

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- If w contains no adjacent blocks of same size and same sum, we say that it is **additive square-free**.
- Do any additive square-free words exist?
- As it turns out, this question is equivalent to our question about double 3-term AP's.

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- The method for calculating each $w^*(r, k)$ is to consider an r-coloring of $[1, n]$ with no double k-AP's, and extend it to $[1, (n+1)]$. We do this as many times as we can.

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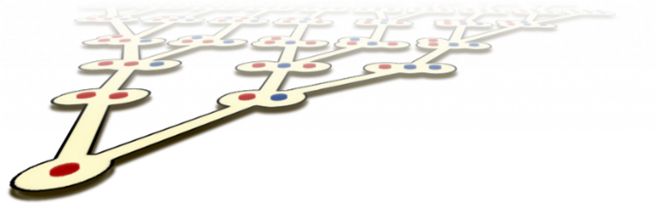
- Abstractly, we define a recursion tree and try to compute its height.



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- Computationally, searching for double AP's is actually easier than searching for ordinary AP's, but the space to search is much larger.
- So, if these numbers are computationally inaccessible, and no good analytic bounds have been found, where can we go from here? One strategy is to restrict the problem, and explore.
- We do this with the utility `RamseyScript`, developed by our group at IRMACS and freely available online.

- For example, let's focus our attention on 3-term AP's and 3-colorings, and restrict the spacing between elements of each color.

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- This coloring has maximal gap size 3 between green elements and maximal gap size 4 between red and blue elements.

We have the following results.

		Max Green Gaps				
		3	4	5	6	7+
Max Blue Gaps	3	22				
	4	31	31			
	5	33	38	43		
	6	33	49	44	45	
	7	33	49	46	46	46
	8+	33	49	46	46	47

Max Red Gap 3

		Max Green Gaps					
		4	5	6	7	8	9+
Max Blue Gaps	4	39					
	5	49	63				
	6	56	79	91			
	7	76	96	>105	>121		
	8	81	96	>114	>121	>130	
	9	81	96	>114	>133	>130	>131
	10	81	96	>114	>133	>135	>135
	11+	81	97	>114	>133	>135	>135

Max Red Gap 4

		Max Green			
		5	6	7	8+
Max Blue	5	100			
	6	> 113	> 133		
	7	?	?	?	
	8+	?	?	?	?

Max Red Gap 5

Alternately, let's consider 2-colorings, and vary the AP length.

		Red	
		2	3
Blue	2	7	
	3	11	17

Double 3-AP's

		Red		
		2	3	4+
Blue	2	11		
	3	22	> 176	
	4+	22	> 2690	> 3573

Double 4-AP's

		Red			
		2	3	4	5+
Blue	2	15			
	3	37	> 131000		
	4	> 25503	?	?	
	5+	> 33366	?	?	?

Double 5-AP's

Based on this evidence, I propose the following:

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- $w^*(2, 3)$ and $w^*(3, 3)$ exist; i.e., there is some number N such that every 2-coloring of $[1, N]$ contains a double 3-AP, and some M such that every 3-coloring of $[1, M]$ contains a double 3-AP.

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- $w^*(2,3)$ and $w^*(3,3)$ exist; i.e., there is some number N such that every 2-coloring of $[1, N]$ contains a double 3-AP, and some M such that every 3-coloring of $[1, M]$ contains a double 3-AP.
- $w^*(r,k)$ does not exist for any greater r or k.

Thank You

Andrew Poelstra <asp11@sfu.ca>