Diversities and Conformities: Generalizing Uniform Convergence

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1 Definition

Definition 1. Let (X, δ) be a diversity, $\{x_n\}$ a sequence of elements of X. We say that $\{x_n\}$ *converges to a limit* x if

$$\lim_{N\to\infty} \sup_{i_1,\ldots,i_n\geq N} \delta(\{x, x_{i_1}, \ldots, x_{i_n}\}) = 0$$

Definition 2. Let (X, δ) be a diversity, $\{x_n\}$ a sequence of elements of X. We say that $\{x_n\}$ is a *Cauchy sequence* if

$$\lim_{N\to\infty} \sup_{i_1,\ldots,i_n\geq N} \delta(\{x_{i_1}, \ldots, x_{i_n}\}) = 0$$

Definition 3. We call a diversity *complete* if every Cauchy sequence converges.

Sanity Check 1. Every convergent sequence is Cauchy.

Proof. Monotonicity. □

2 Metrics and Diversities

Theorem 1. Let (X, δ) be a diversity, d its induced metric. If (X, d) is a complete metric space, then (X, δ) is a complete diversity.

Proof. Suppose that (X, d) is complete. Let $\{x_n\}$ be a Cauchy sequence in (X, δ). Then it is also Cauchy in (X, d), and therefore converges to some element x. We claim that $x_n \to x$ in (X, δ). To this end, let $\epsilon > 0$. Then there exists N such that:

- $d(x_n, x) < \epsilon$ for all $n > N$ (since $x_n \to x$ in (X, d))
- $\delta(\{x_{n_1}, x_{n_2}, \ldots, x_{n_m}\}) < \epsilon$ for all $n_i > N$ (since $\{x_n\}$ is Cauchy in (X, δ).

Therefore, for all $n_1, \ldots, n_m > N$,

$$\begin{aligned}\delta(\{x, x_{n_1}, \ldots, x_{n_m}\}) &\leq \delta(\{x, x_{n_1}\}) + \delta(\{x_{n_1}, \ldots, x_{n_m}\}) \\ &= d(x, x_{n_1}) + \delta(\{x_{n_1}, \ldots, x_{n_m}\}) < 2\epsilon\end{aligned}$$

i.e., $x_n \to x$ in (X, δ). □

Lemma 1. Let (X, d) be a metric space, $\{x_n\}$ a Cauchy sequence in X. If there exists some subsequence $\{x_{i_n}\}$ such that $x_{i_n} \to x$, then $x_n \to x$.

Proof. Let $\epsilon > 0$. Then $d(x_n, x) \leq d(x_n, x_{i_m}) + d(x_{i_m}, x) < 2\epsilon$ for m, n large enough. □

Lemma 2. Let (X,δ) be a diversity, d its induced metric. Let $\{x_n\}$ be Cauchy in (X,d). Then it has a subsequence that is Cauchy in (X,δ).

Proof. Define the subsequence $\{x_{n_i}\}$ by

$$n_i = \min\{n : d(x_n, x_m) < 2^{-i} \text{ for all } m \geq n\}$$

Then given $\epsilon > 0$, choose N such that $2^{1-N} < \epsilon$. Then for all $i_1 \leq i_2 \leq \cdots \leq i_m$ greater than N,

$$\begin{aligned}
\delta(\{x_{i_1}, \ldots, x_{i_m}\}) &\leq \delta(\{x_{i_1}, x_{i_2}\}) + \cdots + \delta(\{x_{i_{m-1}}, x_{i_m}\}) \\
&< 1/2^{i_1} + \cdots + 1/2^{i_m} \\
&< \sum_{i=N}^{\infty} 1/2^i \\
&= 2^{1-N} < \epsilon
\end{aligned}$$

That is, $\{x_{n_i}\}$ is Cauchy in (X,δ). □

Theorem 2. Let (X,δ) be a diversity, d its induced metric. If (X,δ) is a complete diversity, then (X,d) is a complete metric space.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X,d). Then by Lemma 2 it has a subsequence that is Cauchy in (X,δ), which converges to some element x since the diversity is complete. (It converges in both (X,δ) and (X,d).)

Then by Lemma 1, $x_n \to x$ in the metric; i.e., $\{x_n\}$ converges. □

Putting these two theorems together, we get

Corollary 1. A diversity (X,δ) is complete iff its induced metric is complete.

However, we do not have equivalence of Cauchiness, as we will see.

Theorem 3. There exists a diversity (X,δ,d), and a sequence $\{x_n\}$ in X, which is Cauchy in (X,d) but not in (X,δ).

Proof. Let (X,δ) be the Steiner tree diversity on $\mathbb{R}^2$. Define the sets S_i, $i \geq 2$ by

$$S_i = \left\{ \left(\frac{n+1/2}{i^2}, \frac{m+1/2}{i^2} \right) : 0 \leq n \leq i, 0 \leq m \leq i \right\}$$

That is, we have a sequence of $i \times i$ lattices on the square $[0,1/i] \times [0,1/i]$, and the points in S_i are the midpoints.

Order the points in each S_i somehow, and define the sequence

$$\{x_n\} = S_2 S_3 S_4 \cdots$$

Then $\{x_n\}$ is Cauchy in (X,d), since eventually every pair of points is confined to the square $[0,\epsilon] \times [0,\epsilon]$. However, it is not Cauchy in (X,δ), for the following reason:

Consider a minimum spanning tree connecting all the points of some S_i. Since there are i^2 points in S_i, we must have at least i^2-1 edges. Each edge must have length $\geq 1/i^2$, since that is the minimum spacing between points of S_i. So the total length of the edges is

$$\frac{1}{i^2}\cdot(i^2-1) = 1 - \frac{1}{i^2} \geq \frac{3}{4}$$

for every $i \geq 2$. Thus for any $N > 0$,

$$\sup_{i_1,\ldots,i_n \geq N} \delta(\{x_{i_1},\ldots,x_{i_n}\}) \geq \frac{3}{4}C$$

where C is a constant comparing the length of a minimum spanning tree to that of a minimum Steiner tree, in the plane. □

3 Completion

Theorem 4. Every diversity (X, d) can be embedded in a complete diversity.

Proof. Let $\hat{X}$ be the set of all Cauchy sequences in X. Identify any two sequences $\{x_i\}, \{y_i\}$ which satisfy $\lim_{n\to\infty} \delta(\{x_n, y_n\}) = 0$ (so $\hat{X}$ is actually a set of equivalence classes). Define the function $\hat{\delta}$ from $\mathcal{P}_{\text{fin}}(\hat{X}) \to \mathbb{R}$ by

$$\hat{\delta}(\{\{x_i^1\},\{x_i^2\},\ldots,\{x_i^n\}\}) = \lim_{N\to\infty} \sup_{i_1,\ldots,i_n\geq N} \delta(\{x_{i_i}^1, x_{i_2}^2,\ldots,x_{i_n}^n\})$$

It can then be shown that $(\hat{X},\hat{\delta})$ is a complete diversity, and that (X,δ) can be embedded isodiversically in it by $x \mapsto \{x,x,x,\ldots\}$. The proof is straightforward but tedious. □

4 Some Examples

Example 1. *Let (X, d) be a complete metric space, (X,δ) its diameter diversity. Then (X,δ) is complete.*

Example 2. *The Steiner tree diversity on $\mathbb{R}^2$ is complete. (Its induced metric is the Euclidean one, which is complete.)*

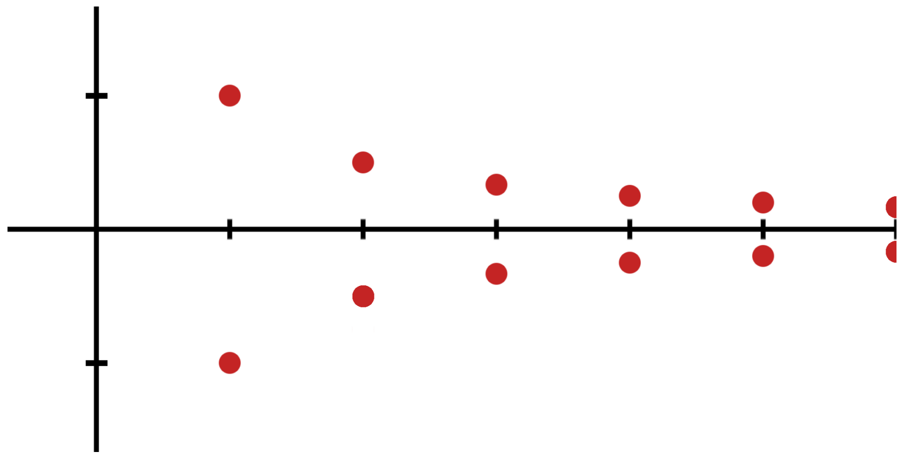
Example 3. *Where diversities behave very differently from their induced metric, the reason is that we can find sets A such that $\delta(A)$ is very large, while $\delta(\{a,b\})$ is small for any $a, b \in A$. We might ask how extreme this effect can be.*

Specifically, does a diversity (X,δ) exist such that?:

- *For all $\epsilon > 0$, there exists $x, y \in X$ such that $\delta(\{x,y\}) < \epsilon$.*
- *There exists some constant C such that for all $Z \subset X$, $\delta(Z) \geq C|Z|$.*

Yes. Consider the euclidean diameter diversity on the set

$$X = \{(n, 1/n) : n \in \mathbb{N}\} \cup \{(n, -1/n) : n \in \mathbb{N}\}$$



For any $\epsilon > 0$, we can find $n > \frac{1}{2\epsilon}$, and the points $(n, 1/n)$ and $(n, -1/n)$ will be within ϵ of each other. However, it is clear that any finite set A will have diameter $\geq |A|/2$.

However, this pathology will not affect our results on convergence, since no such diversity can have a limit point (defined as a point x, with a sequence $\{x_n\}$ such that $x_i \neq x$ for all i, but $x_n \to x$). The reason is simple: for a sequence $\{x_n\}$ to converge in this diversity, it would have to be eventually constant. So "$x_i \neq x$ for every i" is incompatible with "$x_n \to x$".

5 Fixed Points

Definition 4. Let (X, δ) be a diversity, $T : X \to X$ a function such that

$$\delta(T(A)) \leq k\delta(A)$$

for all finite $A \subset X$, some $k \in (0, 1)$. We call T a *contraction mapping* with *Lipschitz constant k.*

Theorem 5. Let (X, δ) be a complete diversity, $T : X \to X$ a contraction mapping with Lipschitz constant k. Then there exists a unique point $x_0 \in X$ such that $T^n(x) \to x_0$ as $n \to \infty$, for all $x \in X$.

Proof. Let $N \in \mathbb{N}$. Then for all $i_1, i_2, \ldots, i_n > N$,

$$\begin{aligned}
&\delta(\{T^{N+i_1}(x), T^{N+i_2}(x), \ldots, T^{N+i_n}(x)\}) \\
&\qquad \leq \delta(\{T^{N+i_1}(x), T^{N+i_2}(x)\}) + \cdots + \delta(\{T^{N+i_{n-1}}(x), T^{N+i_n}(x)\}) \\
&\qquad \leq (k^{N+i_1} + k^{N+i_2} + \cdots + k^{N+i_n})\delta(\{x, T(x)\}) \\
&\qquad \leq \frac{k^N}{1-k}\delta(\{x, T(x)\})
\end{aligned}$$

So by taking N large enough, we can force $\delta(\{T^{N+i_1}(x), \ldots, T^{N+i_n}(x)\})$ as small as we like, and the sequence is Cauchy. Then since (X, δ) is complete, it converges to some fixed point x_0.

As for uniqueness, let y_0 be another fixed point. Then

$$\delta(\{x_0, y_0\}) = \delta(\{T(x_0), T(y_0)\}) \leq k\delta(\{x_0, y_0\})$$

So $\delta(\{x_0, y_0\}) = 0$. □

6 Uniform Spaces and Conformities

Definition 5. Let X be a set, $\mathcal{F}$ a collection of subsets of $\mathcal{P}(X)$. We call $\mathcal{F}$ a *filter* if

- Whenever $X, Y \in \mathcal{F}$, so is $X \cap Y$.
- Whenever $X \in \mathcal{F}$, $Z \supseteq X$, $Z \in \mathcal{F}$.

If $\varnothing \notin \mathcal{F}$, we call $\mathcal{F}$ a *proper filter.*

Definition 6. Let X be a set, $\mathcal{F}$ a collection of subsets of $\mathcal{P}(X)$. We call $\mathcal{F}$ a *filter base* if $\mathcal{F}$ becomes a filter by adding supersets of its elements.

Definition 7. Let X be a set, $\mathcal{U}$ be a collection of subsets of $\mathcal{P}_{\text{fin}}(X)$. We call $\mathcal{U}$ a *conformity* if

- $\mathcal{U}$ is a filter on $\mathcal{P}_{\text{fin}}(X)$.
- For all $U \in \mathcal{U}$, all $x \in X$, $\{x\} \in U$.
- If $a \in U$, $b \subseteq a$, then $b \in U$.
- For all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that

$$V \circ V := \{u \cup v : u, v \in V \text{ and } u \cap v \neq \varnothing\} \subseteq U$$

The pair $(X, \mathcal{U})$ is called a *uniform space.*

This definition corresponds to that of a *uniformity* from metric space theory. The name *conformity* comes from the question, "how does one make a uniformity from a diversity?".

Definition 8. By a *base* for a conformity, we mean a filter base.

Note. Composition, as defined above, is commutative, so that $V \circ V \circ V$ is unambiguous.

Lemma 3. Let $(X, \mathcal{U})$ be a conformity. Then for any $U \in \mathcal{U}$, $n \in \mathbb{N}$, there exists $V \in \mathcal{U}$ such that

$$V \circ V \circ V \subseteq U$$

Proof. Choose V' so that $V' \circ V' \subseteq U$, then V such that $V \circ V \subseteq V'$. Then

$$V \circ V \circ V \subseteq (V \circ V) \circ (V \circ V) = V' \circ V' \subseteq U$$

□

There is a natural way to create conformities from diversities; let (X, δ) be a diversity. Then let

$$\mathcal{U}_{\text{base}} = \{\{x \subset X : |x| < \infty \text{ and } \delta(X) < \epsilon\} : \epsilon > 0\}$$

It is then easily verified that

$$\mathcal{U} = \{U : \exists V \in \mathcal{U}_{\text{base}} \text{ such that } V \subseteq U\}$$

is a diversity. (The last axiom is satisfied by the triangle inequality.)

We can also create conformities from multiple diversities; the generated filter is guaranteed to be proper since every base element contains the diagonal. (So every pair of base elements will have a nonempty intersection.)

6.1 Uniform Continuity

Definition 9. Let (X, δ_X), (Y, δ_Y) be diversities, $f : X \to Y$. We say f is *continuous at a point* x if for all $\epsilon > 0$, there exists some $d = d(x) > 0$ such that whenever $x \in A$, $\delta_X(A) < d$, then $\delta_Y(f(A)) < \epsilon$.

If f is continuous at every point $x \in \Omega$, we say f is *continuous on* Ω.

Definition 10. We say f is *uniformly continuous* if for all $\epsilon > 0$, there exists $d > 0$ such that $\delta_X(A) < d \implies \delta_Y(f(A)) < \epsilon$.

Sanity Check 2. Every uniformly continuous function is continuous.

Theorem 6. Let (X, δ_X), (Y, δ_Y) be diversities, $\mathcal{U}_X$ and $\mathcal{U}_Y$ their conformities. Let $f : X \to Y$. Then f is uniformly continuous iff for all $U \in \mathcal{U}_Y$, $\{f^{-1}(u) : u \in U\} \in \mathcal{U}_X$.

This theorem lets us define uniform continuity purely in terms of conformities, without reference to any underlying diversity.

6.2 Diversification

Lemma 4. Let $(X, \mathcal{U})$ have a countable base. Then it has a countable base $\{U_n\}$ satisfying $U_0 = \mathcal{P}_{\text{fin}}(X)$, $U_i \circ U_i \circ U_i \subseteq U_{i-1}$ for all i.

Proof. Let $\{V_n\}$ be a countable base for $\mathcal{U}$. Define $W_0 = \mathcal{P}_{\text{fin}}(X)$, $W_n = V_n \cap W_{n-1}$. Then $\{W_n\}$ is a nested countable base. Finally, choose $\{U_n\}$ as $U_i = W_{n_i}$, where n_i are chosen inductively as $n_0 = 0$, then $W_{n_i} \circ W_{n_i} \circ W_{n_i} \subseteq W_{n_{i-1}}$. (See Lemma 3.) □

Theorem 7. Let $(X, \mathcal{U})$ be a conformity, $\{U_n\}$ a sequence in $\mathcal{U}$ satisfying $U_0 = \mathcal{P}_{\text{fin}}(X)$, $U_i \circ U_i \circ U_i \subseteq U_{i-1}$ for all $i > 0$. Let $\mathcal{V}$ be the conformity generated by these sets.

Then there exists a pseudodiversity δ on X which generates $\mathcal{V}$.

Proof. We define δ' on $\mathcal{P}_{\text{fin}}$ as follows:

$$\delta'(A) = \begin{cases} 0 & A \in U_n \text{ for all } n \\ 2^{-k} & A \in U_n \text{ for } 0 \leq n \leq k, \text{ but } A \notin U_{k+1} \end{cases}$$

Notice that for $k \geq 0$,

$$\delta'^{-1}([0, 2^{-k}]) = U_k \tag{1}$$

Also note that δ' is monotonic, since if $A \subseteq B$, then A has to be in every U_k that B is. (This is an axiom for $\mathcal{U}$.)

Now, define a **chain** as a finite sequence $\{A_i\}_1^n$ with $A_i \cap A_{i+1} \neq \varnothing$ for $i = 1, \ldots, (n-1)$. Define a **cycle** to be a chain such that $A_1 \cap A_n \neq \varnothing$.

Next, define

$$\bar{\delta}(A) = \inf_{\text{chains covering } A} \sum_{i=1}^{n} \delta'(A_i) \tag{2}$$

$$\delta(A) = \inf_{\text{cycles covering } A} \sum_{i=1}^{n} \delta'(A_i) \tag{3}$$

Also, let $\delta(\varnothing) = \bar{\delta}(\varnothing) = 0$.

There are three stages to our proof:

1. First, we show that δ is a pseudodiversity. We notice that for any singleton $\{x\}$, $\{x\}$ is in every member of $\mathcal{U}$, so that $\delta'(\{x\}) = 0$. Then since $\{x\}$ forms a single-element chain covering itself, $\delta(\{x\}) = 0$.

   Also, the triangle equality holds: let $\epsilon > 0$, $A, C \in \mathcal{P}_{\text{fin}}(X)$ and $B \in \mathcal{P}_{\text{fin}}(X)$ be nonempty. Choose cycles $\{A_i\}_1^n$ and $\{B_i\}_1^m$ which cover $A \cup B$ and $B \cup C$, respectively, and for which

   $$\sum_{i=1}^{n} \delta'(A_i) \leq \delta(A \cup B) + \epsilon \qquad \text{and} \qquad \sum_{i=1}^{m} \delta'(B_i) \leq \delta(B \cup C) + \epsilon$$

   Then $\{A_i\}_1^n \cup \{B_i\}_1^m$ forms a cycle (after reordering) which covers $A \cup C$, so that

   $$\delta(A \cup C) \leq \sum_{i=1}^{n} \delta'(A_i) + \sum_{i=1}^{m} \delta'(B_i) \leq \delta(A \cup B) + \delta(B \cup C) + 2\epsilon$$

   That ϵ is arbitrary gives the result.

2. Next, we claim that

   $$\delta \leq \bar{\delta} \leq 2\delta \tag{4}$$

   This follows easily since

   - Every cycle is a chain, so $\delta \leq \bar{\delta}$.
   - If $\{A_1, \ldots, A_{n-1}, A_n\}$ is a chain, then $\{A_1, \ldots, A_{n-1}, A_n, A_{n-1}, \ldots, A_1\}$ is a cycle — and the sum of δ' over this cycle is less than twice the sum of δ' over the

original chain. It follows that $\bar{\delta} \leq 2\delta$.

3. Finally, we claim that $\bar{\delta}$ satisfies

$$\frac{1}{2}\delta' \leq \bar{\delta} \leq \delta'$$

which will give us that

$$\delta'^{-1}([0,x]) \subseteq \bar{\delta}^{-1}([0,x]) \subseteq \delta'^{-1}([0,2x]) \tag{5}$$

Putting (1) and (5) together gives us

$$U_k = \delta'^{-1}([0,2^{-k}]) \subseteq \bar{\delta}^{-1}([0,x]) \subseteq \delta'^{-1}([0,2^{-K}]) = U_K$$

for $2^{-k} < x < 2^{-K-1}$. Thus $\bar{\delta}^{-1}([0,x])$ is a base for $\mathcal{U}$, and by (4), so is $\delta^{-1}([0,x])$. In other words, δ generates $\mathcal{V}$.

To prove this last step, we first note that $\bar{\delta} \leq \delta'$ trivially. To show $\bar{\delta} \geq \delta'/2$, choose $A \in \mathcal{P}_{\text{fin}}(X)$. Our strategy is to induct on the greatest integer N such that $\bar{\delta}(A) < 2^{-N}$. The case $N = 0$ is trivial, since $\delta' \leq 1$, so $\bar{\delta}(A) > 1/2 \geq \delta'(A)/2$. (For the same reason, the case $\bar{\delta}(A) = 1$, which is not covered by the induction, is trivial.)

For $N > 0$, choose $\epsilon \in \big(0, 2^{-N} - \bar{\delta}(A)\big)$ and a chain $\{A_i\}_1^n$ such that

$$\sum_{i=1}^{n} \delta'(A_i) = \bar{\delta}(A) + \epsilon < 2^{-N} \tag{6}$$

If $n = 1$, our sum is simply $\delta'(A_1)$, so we have

$$\delta'(A) \leq \delta'(A_1) < 2^{-N} < 2\bar{\delta}(A)$$

Otherwise, there is $k < n$ such that

$$\sum_{i=1}^{k-1} \delta'(A_i) \leq \frac{\bar{\delta}(A)}{2} \text{ and } \sum_{i=k+1}^{n} \delta'(A_i) \leq \frac{\bar{\delta}(A)}{2} \tag{7}$$

Since $\{A_i\}_1^{k-1}$ and $\{A_i\}_{k+1}^n$ are chains whose sum under δ' is less than half that of $\{A_i\}_1^n$, the inductive hypothesis applies to them and we may write

$$\begin{aligned}
\delta'(A_1 \cup \cdots \cup A_{k-1}) &\leq 2\bar{\delta}(A_1 \cup \cdots \cup A_{k-1}) && \text{inductive hypothesis} \\
&\leq 2\sum_{i=1}^{k-1} \delta'(A_i) && \text{definition of } \bar{\delta} \\
&\leq \bar{\delta}(A) && \text{by (7)} \\
&< 2^{-N}
\end{aligned}$$

An identical argument gives that $\delta'(A_{k+1} \cup \cdots \cup A_n) < 2^{-N}$, and $\delta'(A_k) < 2^{-N}$ by (6). So

$$(A_1 \cup \cdots \cup A_{k-1}) \in U_{N+1} \text{ and } A_k \in U_{N+1} \text{ and } (A_{k+1} \cup \cdots \cup A_n) \in U_{N+1}$$

Our double-composition hypothesis gives

$$(A_1 \cup \cdots \cup A_{k-1}) \cup A_k \cup (A_{k+1} \cup \cdots \cup A_n) \in U_N$$

And we are done!:

$$\delta'(A) \leq \delta'(A_1 \cup \cdots \cup A_n) \leq 2^{-N} \leq 2\bar{\delta}(A)$$

□

Corollary 2. If $(X,\mathcal{U})$ has a countable base, then there exists a pseudodiversity δ which generates $\mathcal{U}$.

The converse of the above theorem is trivially true.

7 Power Conformities

Definition 11. Let $(X,\mathcal{U})$ be a conformity. We define the *power conformity* $\mathcal{U}^P$ as the conformity on $\mathcal{P}_{\text{fin}}(X)$ generated by sets of the form

$$U_u = \left\{ \{A_1, \ldots, A_n\} : A_i \not\subseteq A_j, \{a_1, \ldots, a_m\} \in u, a_i \in \bigcup_{i=1}^{n} A_i \setminus \bigcap_{i=1}^{n} A_i \right\} \tag{8}$$

for all $u \in \mathcal{U}$.

Our first order of business is to show that diversities are uniformly continuous from their own power conformity to $\mathbb{R}$:

Theorem 8. Let $(X,\mathcal{U})$ be a conformity. A psuedodiversity δ is uniformly continuous from the power conformity $\mathcal{U}^P$ to $(\mathbb{R}, \text{diam})$ iff the sets $V_\epsilon = \{A : \delta(A) < \epsilon\}$ are in $\mathcal{U}$ for every $\epsilon > 0$.

Proof. First, suppose that every V_ϵ is in $\mathcal{U}$. For each $\epsilon > 0$, the set

$$U_\epsilon = \left\{ \{A_1, \ldots, A_n\} : A_i \not\subseteq A_j, \delta(\{a_1, \ldots, a_m\}) < \epsilon, a_i \in \bigcup_{i=1}^{n} A_i \setminus \bigcap_{i=1}^{n} A_i \right\}$$

is in $\mathcal{U}^P$. (Notice it has the form of (8). with $u = V_\epsilon$.) Then for each $\{A, B\} \in U_\epsilon$, we have that $\delta(A \Delta B) < \epsilon$. Since $B \not\subseteq A$, $(B \setminus A) \neq \varnothing$ and we have

$$\begin{aligned}
\delta(A) &= \delta((A \setminus B) \cup (A \cap B)) \\
&\leq \delta\left((A \setminus B) \cup (B \setminus A)\right) + \delta\left((B \setminus A) \cup (A \cap B)\right) \\
&= \delta(A \Delta B) + \delta(B) \\
&< \delta(B) + \epsilon
\end{aligned}$$

Therefore $\delta(A) - \delta(B) < \epsilon$, which shows that δ is uniformly continuous.

Conversely, suppose δ is uniformly continuous. Then for every $\epsilon > 0$, there exists some $u \in \mathcal{U}$ such that every $\{A_1, \ldots, A_n\} \in U_u$ satisfies $\sup_{i,j} |\delta(A_i) - \delta(A_j)| < \epsilon$. In particular, for all $A \neq X \in u$, we can find $x \in X$, $x \notin A$, so the set $\{A, \{x\}\}$ is in U_u, which implies $|\delta(A)| < \epsilon$; i.e., $u \subseteq V_\epsilon$.

(If $X \in u$, then (a) X is a singleton and $\delta(X) = 0 < \epsilon$, or (b) there exist two overlapping sets Y and Z such that $Y \cup Z = X$; then by the triangle inequality $\delta(X) < 2\epsilon$. (So if $\delta(X) > 0$, we can choose a small enough ϵ to avoid this technicality entirely.))

So we have V_ϵ or $V_{2\epsilon}$ in $\mathcal{U}$, which is the result. □

This gives us the main result of this section:

Corollary 3. Every conformity $(X,\mathcal{U})$ is generated by the set of all pseudodiversities which are uniformly continuous from $\mathcal{U}$ to the diameter diversity on $\mathbb{R}$.

Proof. Let $\mathcal{V}$ be the conformity generated by all uniformly continuous pseudodiversities. The previous result implies that every set in $\mathcal{V}$ is also in $\mathcal{U}$; the diversification result of the last article implies that every set in $\mathcal{U}$ is also in $\mathcal{V}$. □